ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT®
J. Math. Anal. Appl. 296 (2004) 695–707
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Spaces $M(\mathcal{D}_p)$, $\mathcal{B}_n^p$, and Q_p ☆

Pengyan Hu

Department of Mathematics, Normal College of Shenzhen University, Shenzhen, Guangdong, 518060, PR China

Received 15 December 2003
Available online 2 July 2004
Submitted by T. Akahori

Abstract

In this paper, the relations between $\mathcal{D}_p$ and $\mathcal{B}_n^p$, $M(\mathcal{D}_p)$ and $\mathcal{B}_n^p$ are discussed by means of high order radial derivative and random series. The main results of this paper are:

(1) Let $-\infty < p < n+1$,
  (i) if $0 < q < n - p/2$, then $\mathcal{B}_n^q \subset \mathcal{D}_p$;
  (ii) if $n - p/2 \leqslant q < (3n-p)/2$, then $\mathcal{B}_n^q$ and $\mathcal{D}_p$ cannot included each other;
  (iii) if $q \geqslant (3n-p)/2$, then $\mathcal{D}_p \subset \mathcal{B}_n^q$.
(2) Let $0 < p \leqslant n$,
  (i) if $0 < q < n - p/2$, then $\mathcal{B}_n^q \subset M(\mathcal{D}_p)$;
  (ii) if $n - p/2 \leqslant q < (3n-p)/2$, then $\mathcal{B}_n^q$ and $M(\mathcal{D}_p)$ cannot included each other;
  (iii) if $q \geqslant (3n-p)/2$, then $M(\mathcal{D}_p) \subset \mathcal{B}_n^q$.
(3) Let $(n-1)/n < p \leqslant 1$, then $M(\mathcal{D}_{n(1-p)}) \subset Q_p$.

Keywords: Multiplication operator; Bloch type space $\mathcal{B}_n^p$; Q_p space; Random series

1. Introduction

Let B be the unit ball in the complex vector space $\mathbb{C}^n$ with dimension n $(n \geqslant 1)$, S be the boundary of B, and ν and σ be the Lebesgue measures on B and S, respectively. Denote

☆ Supported by Natural National Science Foundation of China, 10101013.
E-mail address: pyhu@szu.edu.cn.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.04.042

$d\lambda(z) = d\nu(z)/((1-|z|^2)^{n+1})$ the invariant measure on B. Let $H(B)$ be the collection of the holomorphic functions with domain B.

Let $a \in B$, denote $\varphi_a(z)$ the holomorphic automorphism [1] on B, which satisfies $\varphi_a(0) = a$, $\varphi_a(a) = 0$, and

$$1 - \left|\varphi_a(z)\right|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z, a\rangle|^2}, \tag{1}$$

for $z \in B$.

Denote $\nabla f = (\partial f/\partial z_1, \ldots, \partial f/\partial z_n)$ the complex gradient of f, and $\mathcal{R} f(z) = \sum_{j=1}^n z_j(\partial f/\partial z_j)(z) = \sum_{j=1}^n z_j D_j f(z)$ the radial derivative of f. Let $\tilde{\Delta} f(z) = \Delta(f \circ \varphi_z)(0)$ be the invariant Laplacian of f, here Δ is the ordinary Laplacian. Let

$$\left|\tilde{\nabla} f(z)\right|^2 = \frac{1}{n+1}\left|\nabla(f \circ \varphi_z)(0)\right|^2$$

be the invariant gradient of f. The Green function related to the invariant Laplacian $\tilde{\Delta}$ is $G(z,a) = g(\varphi_a(z))$, here

$$g(z) = \frac{n+1}{2n} \int_{|z|}^{1} \left(1-t^2\right)^{n-1} t^{-2n+1}\, dt.$$

For $0 < p < \infty$, define the function space Q_p and p-Bloch space $\mathcal{B}^p$ as follows [2]:

$$Q_p = \left\{ f \in H(B)\colon \sup_{a \in B} \int_B \left|\tilde{\nabla} f(z)\right|^2 G^p(z,a)\, d\lambda(z) < \infty \right\},$$

$$\mathcal{B}^p = \left\{ f \in H(B)\colon \sup_{z \in B} \left(1-|z|^2\right)^p \left|\nabla f(z)\right| < \infty \right\}.$$

It is easy to verify that $f \in \mathcal{B}^p$ if and only if $\sup_{z\in B}(1-|z|^2)^p|\mathcal{R} f(z)| < \infty$.

Let $f(z) = \sum_{\alpha \geqslant 0} a_\alpha z^\alpha \in H(B)$, the radial derivative of order n of f, $\mathcal{R}^{(n)} f$, is defined as

$$\mathcal{R}^{(n)} f(z) = \sum_{\alpha \geqslant 0} |\alpha|^n a_\alpha z^\alpha.$$

Obviously, $\mathcal{R}^{(1)} f(z) = \mathcal{R} f(z)$. Define the Bloch type space $\mathcal{B}_n^p$ as

$$\mathcal{B}_n^p = \left\{ f \in H(B)\colon \mathcal{R}^{(n-1)} f \in \mathcal{B}^p \right\},$$

then $\mathcal{B}_1^p = \mathcal{B}^p$, and $\mathcal{B}_n^p$ is increase of p. If $p > n-1$, $\mathcal{B}_n^p = \mathcal{B}^{p-n+1}$. For $p \in \mathbb{R}$, Dirichlet type space $\mathcal{D}_p$ are defined by [3]

$$\mathcal{D}_p = \left\{ f(z) = \sum_{\alpha \geqslant 0} a_\alpha z^\alpha \in H(B)\colon \sum_{\alpha \geqslant 0} \left(n + |\alpha|\right)^p |a_\alpha|^2 \omega_\alpha < \infty \right\},$$

where [1]

$$\omega_\alpha = \int_S \left|\zeta^\alpha\right|^2 d\sigma(\zeta) = \frac{(n-1)!\alpha!}{(n+|\alpha|-1)!}.$$

These spaces include Hardy space $H^2(B)$ $(p=0)$, Bergman space $L_a^2(B)(p=-1)$, and Dirichlet space $\mathcal{D}(p=n)$.

Let X and Y be function spaces, we call the function ϕ to be a multiplication operator from X to Y, provided that $\phi f \in Y$ for any $f \in X$. The collection of the multiplication operator from X to Y is denoted by $M(X, Y)$. Specially, $M(X, X)$ is denoted by $M(X)$ simply.

In [3], the authors give the characterizations of the multiplication operators on Dirichlet type spaces by taking advantage of the Taylor coefficients of the holomorphic functions. Specially, it is proved that $M(\mathcal{D}_p)=\mathcal{D}_p$ when $p>n$. This result implies that $\mathcal{D}_p$ is an algebra when $p>n$. The random series are used to characterize the multiplication operators in [4]. In [5], the Dirichlet type spaces are studied more deeply, and a characterization of *BMO* type for these spaces is given. The discussion about the function Q_p and $\mathcal{B}^p$ and their relations are given in [2,6,7]. In this paper, the relations between $\mathcal{D}_p$ and $\mathcal{B}_n^p$, $M(\mathcal{D}_p)$ and $\mathcal{B}_n^p$, and $M(\mathcal{D}_p)$ and Q_p are given. These results can give more clear understanding about the function spaces mentioned above. From one point, they can regard as a type of characterizations of the spaces $M(\mathcal{D}_p)$. Our main results are the following theorems.

Theorem 1. *Let* $-\infty<p<n+1$.

(i) *If* $0<q<n-p/2$, *then* $\mathcal{B}_n^q \subset \mathcal{D}_p$.
(ii) *If* $n-p/2 \leqslant q<(3n-p)/2$, *then* $\mathcal{B}_n^q$ *and* $\mathcal{D}_p$ *cannot include each other.*
(iii) *If* $q \geqslant (3n-p)/2$, *then* $\mathcal{D}_p \subset \mathcal{B}_n^q$.

Theorem 2. *Let* $0<p\leqslant n$.

(i) *If* $0<q<n-p/2$, *then* $\mathcal{B}_n^q \subset M(\mathcal{D}_p)$.
(ii) *If* $n-p/2 \leqslant q<(3n-p)/2$, *then* $\mathcal{B}_n^q$ *and* $M(\mathcal{D}_p)$ *cannot include each other.*
(iii) *If* $q \geqslant (3n-p)/2$, *then* $M(\mathcal{D}_p) \subset \mathcal{B}_n^q$.

Theorem 3. *Let* $(n-1)/n<p\leqslant 1$, *then* $M(\mathcal{D}_{n(1-p)}) \subset Q_p$.

In the sequel, the symbols are the same as in [1] if their meanings are not indicated clearly. C and C_j denote constants, and they can take different values in different appearances. The symbol "$A \approx B$" implies that there are constants C_1 and C_2 such that $C_1 B \leqslant A \leqslant C_2 B$.

2. $\mathcal{B}_n^p$ and $\mathcal{D}_p$

In this section, the proof of Theorem 1 will be given. First, we give some lemmas.

Lemma 1 [8]. *For* $p<n+1$, *the following statements are equivalent*:

(i) $f \in \mathcal{D}_p$;
(ii) $\int_B |\mathcal{R}^{(n)} f(z)|^2 (1-|z|^2)^{2n-1-p}\, d\nu(z)<\infty$;

(iii) $\int_B |\nabla(\mathcal{R}^{(n-1)} f(z))|^2 (1-|z|^2)^{2n-1-p}\, d\nu(z) < \infty$.

Next result can be derived by the same method used in [9, Theorem 4.10].

Lemma 2. *Let $n \geqslant 2$, $p \geqslant 0$, $f \in H(B)$. For $\zeta \in S$, let $f_\zeta(\lambda) = f(\lambda\zeta)$. If there exists a constant $M > 0$ such that*

$$\|f_\zeta\|_{\mathcal{B}^p(U)} \leqslant M,$$

for every $\zeta \in S$, then $f \in \mathcal{B}^p$. Here U denotes the unit disc in the complex plane.

Lemma 3. *Let $f \in H(B)$, $0 \leqslant p < \infty$, $0 < r < 1$, $0 < s < \infty$, $n < t < \infty$, then the following statements are equivalent*:

$$\text{(i)}\quad \sup_{z\in B} \left(1-|z|^2\right)^{ps} \left|\nabla f(z)\right|^s < \infty; \tag{2}$$

$$\text{(ii)}\quad \sup_{z\in B} \frac{1}{|E(z,r)|^{1-ps/(n+1)}} \int\limits_{E(z,r)} \left|\nabla f(w)\right|^s d\nu(w) < \infty; \tag{3}$$

$$\text{(iii)}\quad \sup_{z\in B} \int\limits_{E(z,r)} \left|\nabla f(w)\right|^s \left(1-|w|^2\right)^{ps-n-1} d\nu(w) < \infty; \tag{4}$$

$$\text{(iv)}\quad \sup_{z\in B} \int\limits_{B} \left|\nabla f(w)\right|^s \left(1-|w|^2\right)^{ps-n-1} \left(1-\left|\varphi_z(w)\right|^2\right)^t d\nu(w) < \infty, \tag{5}$$

where $|E(z,r)| = \nu(E(z,r))$ and $E(z,r) = \{w \in B\colon |\varphi_z(w)| < r\}$.

Proof. (ii) $\Rightarrow$ (i). Suppose that (3) is valid. If $g(w)$ is a subharmonic function on B, then, for fixed r,

$$g(0) \leqslant C \int\limits_{|w|<r} g(w)\, d\nu(w). \tag{6}$$

Take $g(w) = |D_j f(\varphi_z(w))|^s$ in (6) and use variable transformation in the integration, then

$$\begin{aligned}
\left|D_j f(z)\right|^s &\leqslant C \int\limits_{|w|<r} \left|D_j f\left(\varphi_z(w)\right)\right|^s d\nu(w) \\
&= C \int\limits_{E(z,r)} \left|D_j f(w)\right|^s J_R \varphi_z(w)\, d\nu(w) \\
&= C \int\limits_{E(z,r)} \left|D_j f(w)\right|^s \left(\frac{1-|z|^2}{|1-\langle z, w\rangle|^2}\right)^{n+1} d\nu(w),
\end{aligned}$$

where $J_R\varphi_z$ is the real Jacobian of φ_z. Since $1-|z|^2 \approx 1-|w|^2$ and $|1-\langle z,w\rangle| \approx 1-|w|^2$ as $w \in E(z,r)$, and [1] $|E(z,r)| \approx (1-|z|^2)^{n+1}$, then

$$
\begin{aligned}
\bigl(1-|z|^2\bigr)^{ps}\bigl|D_j f(z)\bigr|^s &\leqslant C\bigl(1-|z|^2\bigr)^{ps-n-1}\int\limits_{E(z,r)}\bigl|D_j f(w)\bigr|^s\,d\nu(w)\\
&= \frac{C}{((1-|z|^2)^{n+1})^{1-ps/(n+1)}}\int\limits_{E(z,r)}\bigl|D_j f(w)\bigr|^s\,d\nu(w)\\
&\leqslant \frac{C}{|E(z,r)|^{1-ps/(n+1)}}\int\limits_{E(z,r)}\bigl|D_j f(w)\bigr|^s\,d\nu(w). \qquad (7)
\end{aligned}
$$

So it follows by (7) that

$$
\begin{aligned}
\bigl(1-|z|^2\bigr)^{ps}\bigl|\nabla f(z)\bigr|^s &\leqslant C\bigl(1-|z|^2\bigr)^{ps}\sum_{j=1}^{n}\bigl|D_j f(z)\bigr|^s\\
&\leqslant \frac{C}{|E(z,r)|^{1-ps/(n+1)}}\int\limits_{E(z,r)}\sum_{j=1}^{n}\bigl|D_j f(w)\bigr|^s\,d\nu(w)\\
&\leqslant \frac{C}{|E(z,r)|^{1-ps/(n+1)}}\int\limits_{E(z,r)}\bigl|\nabla f(w)\bigr|^s\,d\nu(w).
\end{aligned}
$$

This gives (2).

(iii) $\Rightarrow$ (ii). Since $1-|z|^2 \approx 1-|w|^2$ for $w \in E(z,r)$ and $|E(z,r)| \approx (1-|z|^2)^{n+1}$, then

$$
\begin{aligned}
&\frac{1}{|E(z,r)|^{1-ps/(n+1)}}\int\limits_{E(z,r)}\bigl|\nabla f(w)\bigr|^s\,d\nu(w)\\
&\quad\leqslant C\int\limits_{E(z,r)}\bigl|\nabla f(w)\bigr|^s\bigl(1-|w|^2\bigr)^{ps-n-1}\,d\nu(w). \qquad (8)
\end{aligned}
$$

Therefore (8) implies that (3) can be get from (4).

(iv) $\Rightarrow$ (iii). For $t > n$,

$$
\begin{aligned}
&\int\limits_{E(z,r)}\bigl|\nabla f(w)\bigr|^s\bigl(1-|w|^2\bigr)^{ps-n-1}\,d\nu(w)\\
&\quad\leqslant \frac{1}{(1-r^2)^t}\int\limits_{E(z,r)}\bigl|\nabla f(w)\bigr|^s\bigl(1-|w|^2\bigr)^{ps-n-1}\bigl(1-\bigl|\varphi_z(w)\bigr|^2\bigr)^t\,d\nu(w)\\
&\quad\leqslant C\int\limits_{B}\bigl|\nabla f(w)\bigr|^s\bigl(1-|w|^2\bigr)^{ps-n-1}\bigl(1-\bigl|\varphi_z(w)\bigr|^2\bigr)^t\,d\nu(w),
\end{aligned}
$$

then (4) is valid if so is (5).

(i) $\Rightarrow$ (iv). Assume that (2) is true, then there exists a constant $M > 0$ such that

$$
\sup_{z\in B}\bigl(1-|z|^2\bigr)^{ps}\bigl|\nabla f(z)\bigr|^s = M < \infty. \qquad (9)
$$

For $z \in B$, by (1), (9), and [1, Lemma 1.4.10],

$$\begin{aligned}
&\int_B \left|\nabla f(w)\right|^s \left(1-|w|^2\right)^{ps-n-1}\left(1-\left|\varphi_z(w)\right|^2\right)^t \, d\nu(w) \\
&\quad \leqslant M \int_B \frac{(1-|z|^2)^t(1-|w|^2)^{t-n-1}}{|1-\langle w,z\rangle|^{2t}} \, d\nu(w) \\
&\quad = M\left(1-|z|^2\right)^t\left(1-|z|^2\right)^{-t} = CM.
\end{aligned}$$

This gives (5). □

Proof of Theorem 1. (i) For any $f \in \mathcal{B}_n^q$, since $0 < q < n - p/2$, then

$$\begin{aligned}
&\int_B \left|\mathcal{R}^{(n)} f(z)\right|^2 \left(1-|z|^2\right)^{2n-1-p} \, d\nu(z) \\
&\quad = \int_B \left|\mathcal{R}^{(n)} f(z)\right|^2 \left(1-|z|^2\right)^{2q}\left(1-|z|^2\right)^{2n-1-p-2q} \, d\nu(z) \\
&\quad \leqslant \left\|\mathcal{R}^{(n-1)} f\right\|_{\mathcal{B}^q}^2 \int_B \left(1-|z|^2\right)^{2n-1-p-2q} \, d\nu(z) < \infty.
\end{aligned}$$

So, Lemma 1 gives that $f \in \mathcal{D}_p$.

(ii) First we prove that $\mathcal{B}_n^q \not\subset \mathcal{D}_p$. It is well known that [10] there exists a homogenous polynomial of order k, $p_k(\zeta)$, on S for every positive integer k satisfying

$$\|p_k\|_\infty = \sup_{\zeta \in S}\left|p_k(\zeta)\right| = 1, \qquad \|p_k\|_2 = \left(\int_S \left|p_k(\zeta)\right|^2 \, d\sigma(\zeta)\right)^{1/2} \geqslant \sqrt{\pi}\, 2^{-n}.$$

Choose

$$f_1(z) = \sum_{\alpha \geqslant 0} a_\alpha z^\alpha = \sum_{k=1}^{\infty} a_k p_{2^k}(z),$$

where $a_k = 1/(\sqrt{k}\, 2^{kp/2})$. For any $\zeta \in S$ and $\lambda \in U$, let $g_1(\lambda) = f_1(\lambda\zeta)$. Then

$$\mathcal{R}^{(n-1)} g_1(\lambda) = \sum_{k=1}^{\infty} 2^{(n-1)k} a_k p_{2^k}(\zeta)\lambda^{2^k}.$$

Since

$$2^{k(1-n+p/2)} 2^{(n-1)k}|a_k|\left|p_{2^k}(\zeta)\right| \approx 2^{k(1-n+p/2+n-1)} \frac{1}{\sqrt{k}\, 2^{kp/2}} = \frac{1}{\sqrt{k}},$$

then

$$\lim_{k\to\infty} 2^{k(1-n+p/2)} 2^{(n-1)k}|a_k|\left|p_{2^k}(\zeta)\right| = \lim_{k\to\infty} \frac{1}{\sqrt{k}} = 0. \tag{10}$$

So we get [11]

$$\mathcal{R}^{(n-1)}g_1 \in \mathcal{B}_0^{n-p/2}(U) \subset \mathcal{B}^{n-p/2}(U).$$

Therefore, it follows that $\mathcal{R}^{(n-1)}f_1 \in \mathcal{B}^{n-p/2}$ from Lemma 2, that is $f_1 \in \mathcal{B}_n^{n-p/2} \subset \mathcal{B}_n^q$.

On the other hand, since

$$\sum_{\alpha \geqslant 0} \big(n+|\alpha|\big)^p |a_\alpha|^2 \omega_\alpha \geqslant C \sum_{k=1}^{\infty} \frac{2^{kp}}{k2^{kp}} \|p_{2^k}\|_2^2 \geqslant C \sum_{k=1}^{\infty} \frac{1}{k} = \infty,$$

then $f_1 \notin \mathcal{D}_p$. So $\mathcal{B}_n^q \not\subset \mathcal{D}_p$ when $q \geqslant n - p/2$.

Next, we will prove that $\mathcal{D}_p \not\subset \bigcup_{0<q<(3n-p)/2} \mathcal{B}^q$.

Let

$$f_2(z) = \sum_{\alpha \geqslant 0} a_\alpha z^\alpha = \sum_{k=2}^{\infty} a_k z_1^k = g_2(z_1),$$

where $a_k = 1/(k^{(2-n+p)/2}(\log k)^s)$ and $s > 1/2$. Since

$$\sum_{\alpha \geqslant 0} \big(n+|\alpha|\big)^p |a_\alpha|^2 \omega_\alpha \leqslant C \sum_{k=2}^{\infty} \frac{k^p}{k^{2-n+p}(\log k)^{2s} k^{n-1}} = C \sum_{k=2}^{\infty} \frac{1}{k(\log k)^{2s}} < \infty,$$

then $f_2 \in \mathcal{D}_p$.

By the increase of $\mathcal{B}_n^q$ of q, we can assume that $(3n-2-p)/2 < q < (3n-p)/2$, then $p + 2q - 3n + 2 > 0$ and $3n - p - 2q > 0$. Since

$$\mathcal{R}^{(n-1)}g_2(z_1) = \sum_{k=2}^{\infty} k^{n-1} a_k z_1^k,$$

then

$$\begin{aligned}\sum_{k=m}^{2m-1} k^{1-q} k^{n-1} a_k &= \sum_{k=m}^{2m-1} \frac{k^{n-q}}{k^{(2-n+p)/2}(\log k)^s} = \sum_{k=m}^{2m-1} \frac{1}{k^{(p+2q-3n+2)/2}(\log k)^s} \\ &\geqslant C \frac{m^{(3n-p-2q)/2}}{(\log(2m-1))^s} \to \infty \quad (m \to \infty).\end{aligned} \tag{11}$$

So $\mathcal{R}^{(n-1)}g_2 \notin \mathcal{B}^q(U)$ from the proof of (i) $\Rightarrow$ (ii) of [12, Theorem 3.1]. By the same method used in the proof of [13, Theorem 1] we have $\mathcal{R}^{(n-1)}f_2 \notin \mathcal{B}^q$, that is $f_2 \notin \mathcal{B}_n^q$.

(iii) Let $t > n$, for any $z \in B$,

$$\begin{aligned}&\int_B \big|\nabla\big(\mathcal{R}^{(n-1)}f(w)\big)\big|^2 \big(1-|w|^2\big)^{(2(3n-p))/2-n-1} \big(1-|\varphi_z(w)|^2\big)^t \, d\nu(w) \\ &\quad \leqslant \int_B \big|\nabla\big(\mathcal{R}^{(n-1)}f(w)\big)\big|^2 \big(1-|w|^2\big)^{2n-1-p} \, d\nu(w),\end{aligned} \tag{12}$$

then (12), Lemmas 1 and 3 give that $\mathcal{D}_p \subset \mathcal{B}^{(3n-p)/2}$. Therefore, $\mathcal{D}_p \subset \mathcal{B}_n^q$ as $q \geqslant (3n-p)/2$. $\square$

3. $M(\mathcal{D}_p)$, $\mathcal{B}_n^p$ and Q_p

In this section, we will give the proofs of Theorems 2 and 3. These results can be regard as the characterizations of $M(\mathcal{D}_p)$. For $f \in H(B)$, $0 < p < \infty$, $0 \leqslant r < 1$, let

$$M_p(f,r) = \left(\int_S \left| f(r\zeta) \right|^p d\sigma(\zeta) \right)^{1/p} \quad \text{and} \quad M_\infty(f,r) = \sup_{\zeta \in S} \left| f(r\zeta) \right|.$$

First, we will give some lemmas.

Lemma 4 [14]. *Let $f \in H(B)$, $0 < p < \infty$, $\delta \geqslant 0$. If*

$$M_p(r,f) \leqslant C(1-r)^{-\delta}, \tag{13}$$

then there is a constant $K = K(p,\delta)$, such that

$$M_q(r,f) \leqslant KC(1-r)^{n/q-n/p-\delta}, \tag{14}$$

for q, $p < q \leqslant \infty$, and

$$M_q(r,f) = o\big((1-r)^{n/q-n/p}\big) \quad (r \to 1^-), \tag{15}$$

when $\delta = 0$.

Lemma 5. *Let $\phi \in H(B)$ and $0 < p \leqslant n$. If*

$$\int_0^1 M_q^2\big(r, \mathcal{R}^{(n)}\phi\big)(1-r)^{2n-1-p}\,dr < \infty, \tag{16}$$

for some $q > 2n/p$, then $\phi \in M(\mathcal{D}_p)$.

Proof. For any $\varepsilon > 0$, (16) implies that there is an $r_0 > 0$, such that

$$\int_r^1 M_q^2\big(t, \mathcal{R}^{(n)}\phi\big)(1-t)^{2n-1-p}\,dt < \varepsilon,$$

as $1 > r > r_0$. So

$$M_q\big(r, \mathcal{R}^{(n)}\phi\big) = o\big((1-r)^{p/2-n}\big), \quad r \to 1^-,$$

by the monotonicity of $M_q(r, \mathcal{R}^{(n)}\phi)$ with r. By Lemma 4,

$$M_\infty\big(r, \mathcal{R}^{(n)}\phi\big) = o\big((1-r)^{-n/q+p/2-n}\big), \quad r \to 1^-.$$

Therefore,

$$M_\infty\big(r, \mathcal{R}^{(n-j)}\phi\big) = o\big((1-r)^{-n/q+p/2-n+j}\big), \quad r \to 1^-, \tag{17}$$

for the integer j, $j < n + n/q - p/2$. For p and a suitable q, there is an integer k, which satisfies that $-1 < -n/q + p/2 - n + k < 0$. Then (17) for $j = k$ and [1, Theorem 6.4.10] give that $\mathcal{R}^{(n-k-1)}\phi \in H^\infty(B)$. So for $j = k+1, k+2, \ldots, n$, $\mathcal{R}^{(n-j)}\phi \in H^\infty(B)$.

Let $f \in \mathcal{D}_p$, then

$$\begin{aligned}
&\int_B \left|\mathcal{R}^{(n)}(\phi f)(z)\right|^2\left(1-|z|^2\right)^{2n-1-p}\,d\nu(z)\\
&\quad=\int_B \left|\sum_{j=0}^{n}\mathcal{R}^{(n-j)}\phi(z)\mathcal{R}^{(j)}f(z)\right|^2\left(1-|z|^2\right)^{2n-1-p}\,d\nu(z)\\
&\quad\leqslant\sum_{j=0}^{n}C_j\int_B \left|\mathcal{R}^{(n-j)}\phi(z)\right|^2\left|\mathcal{R}^{(j)}f(z)\right|^2\left(1-|z|^2\right)^{2n-1-p}\,d\nu(z)\\
&\quad=C_0\int_B \left|\mathcal{R}^{(n)}\phi(z)\right|^2\left|f(z)\right|^2\left(1-|z|^2\right)^{2n-1-p}\,d\nu(z)\\
&\qquad+\sum_{j=1}^{k}C_j\int_B \left|\mathcal{R}^{(n-j)}\phi(z)\right|^2\left|\mathcal{R}^{(j)}f(z)\right|^2\left(1-|z|^2\right)^{2n-1-p}\,d\nu(z)\\
&\qquad+\sum_{j=k+1}^{n}C_j\int_B \left|\mathcal{R}^{(n-j)}\phi(z)\right|^2\left|\mathcal{R}^{(j)}f(z)\right|^2\left(1-|z|^2\right)^{2n-1-p}\,d\nu(z)\\
&\quad=\mathrm{M}_1+\mathrm{M}_2+\mathrm{M}_3.
\end{aligned}\tag{18}$$

For M_3, since $\mathcal{R}^{(n-j)}\phi \in H^\infty(B)$ as $j=k+1,k+2,\dots,n$, then

$$\mathrm{M}_3\leqslant\sum_{j=k+1}^{n}C_j\int_B \left|\mathcal{R}^{(j)}f(z)\right|^2\left(1-|z|^2\right)^{2n-1-p}\,d\nu(z).\tag{19}$$

But $f \in \mathcal{D}_p$ implies that $f \in \mathcal{D}_{p-2(n-j)}$ for $0 \leqslant j \leqslant n$. Therefore (19) gives $\mathrm{M}_3 < \infty$.

For M_2, choose q large enough, such that $2j - 2n/q > 0$ for j, $1 \leqslant j \leqslant k$. Since $2n/q < p$, then $f \in \mathcal{D}_{2n/q}$ if $f \in \mathcal{D}_p$. So by (17),

$$\begin{aligned}
\mathrm{M}_2&\leqslant\sum_{j=1}^{k}C_j\int_B \left|\mathcal{R}^{(j)}f(z)\right|^2\left(1-|z|^2\right)^{2n-1-p-2n/q+p-2n+2j}\,d\nu(z)\\
&=\sum_{j=1}^{k}C_j\int_B \left|\mathcal{R}^{(j)}f(z)\right|^2\left(1-|z|^2\right)^{2j-1-2n/q}\,d\nu(z)<\infty.
\end{aligned}\tag{20}$$

For M_1, by using integration in polar coordinate, Hölder's inequality for conjugate indices $q/2$ and $q/(q-2)$, and [4, Corollary 2.1],

$$\begin{aligned}
\mathrm{M}_1&=C_0\int_0^1 r^{2n-1}\left(1-r^2\right)^{2n-1-p}\,dr\int_S \left|f(r\zeta)\right|^2\left|\mathcal{R}^{(n)}\phi(r\zeta)\right|^2\,d\sigma(\zeta)\\
&\leqslant C_0\int_0^1 r^{2n-1}\left(1-r^2\right)^{2n-1-p}\,dr\left(\int_S \left|\mathcal{R}^{(n)}\phi(r\zeta)\right|^q\,d\sigma(\zeta)\right)^{2/q}
\end{aligned}$$

$$\cdot \left(\int_S \left| f(r\zeta) \right|^{2q/(q-2)} d\sigma(\zeta) \right)^{(q-2)/q}$$

$$\leqslant C_0 \| f \|_{2q/(q-2)}^2 \int_0^1 M_q^2\bigl(r, \mathcal{R}^{(n)}\phi\bigr)(1-r)^{2n-1-p}\, dr < \infty. \tag{21}$$

It follows that $\phi f \in \mathcal{D}_p$ from (18)–(21) and Lemma 1, that is $\phi \in M(\mathcal{D}_p)$. $\square$

Lemma 6. *For $0 < p \leqslant n$, there exists a positive integer k, such that $0 < p/2 - n + k < 1$, then $\mathrm{Lip}_{p/2-n+k}^{(n-k)} = \mathcal{B}_n^{n-p/2}$. It is said that $f \in \mathrm{Lip}_{p/2-n+k}^{(n-k)}$ if and only if $\mathcal{R}^{(n-k)} f \in \mathrm{Lip}_{p/2-n+k}$.*

Proof. Let $f \in H(B)$. Since [1] $f \in \mathrm{Lip}_{p/2-n+k}^{(n-k)}$ if and only if

$$M_\infty\bigl(r, \mathcal{R}^{(n-k+1)} f\bigr) = O\bigl((1-r)^{p/2-n+k-1}\bigr), \tag{22}$$

and $f \in \mathcal{B}_n^{n-p/2}$ if and only if

$$\sup_{0 \leqslant r < 1} (1-r)^{n-p/2} M_\infty\bigl(r, \mathcal{R}^{(n)} f\bigr) < \infty, \tag{23}$$

then (22) and (23) imply that $\mathrm{Lip}_{p/2-n+k}^{(n-k)} = \mathcal{B}_n^{n-p/2}$. $\square$

Now we are ready to give the proof of Theorem 2.

Proof of Theorem 2. (i) If $f \in \mathcal{B}_n^q$, then

$$\sup_{z \in B} \bigl(1 - |z|^2\bigr)^q \left| \mathcal{R}^{(n)} f(z) \right| < \infty. \tag{24}$$

So

$$\int_0^1 M_s^2\bigl(r, \mathcal{R}^{(n)} f\bigr)(1-r)^{2n-1-p}\, dr$$

$$= \int_0^1 \left(\int_S \left| \mathcal{R}^{(n)} f(r\zeta) \right|^s d\sigma(\zeta) \right)^{2/s} (1-r)^{2n-1-p}\, dr$$

$$\leqslant C \int_0^1 (1-r)^{2n-1-p-2q}\, dr < \infty,$$

for $s > 2n/p$. Thus $f \in M(\mathcal{D}_p)$ by Lemma 5.

(ii) Since $M(\mathcal{D}_p) \subset \mathcal{D}_p$, then Theorem 1(ii) gives that $\mathcal{B}_n^{n-p/2} \not\subset M(\mathcal{D}_p)$. On the other hand, choose $c_\alpha = 1/\log|\alpha|$ in [4, Theorem 1.3], then there is a function

$$f(z) = \sum_{\alpha \geqslant 0} \frac{a_\alpha}{|\alpha|^{n-j}} z^\alpha$$

satisfying

$$\sum_{\alpha \geqslant 0} \bigl(n+|\alpha|\bigr)^{p-2(n-j)} |a_\alpha|^2 = \sum_{\alpha \geqslant 0} c_\alpha \bigl(n+|\alpha|\bigr)^{p-2(n-j)} |a_\alpha|^2 \log|\alpha| < \infty, \tag{25}$$

and $\mathcal{R}^{(n-j)} f_\omega \notin \mathrm{Lip}^{(n-j)}_{p/2-n+j}$ almost surely, where j is a positive integer satisfying $0 \leqslant p-2(n-j) < 2$, and f_ω is the randomization [4] of f. So it follows that $f_\omega \notin \mathcal{B}_n^{n-p/2}$ almost surely by Lemma 6.

But,

$$\sum_{\alpha \geqslant 0} \bigl(n+|\alpha|\bigr)^{p} \frac{|a_\alpha|^2}{|\alpha|^{2(n-j)}} \xi^2_{\alpha,q} \leqslant \sum_{\alpha \geqslant 0} \bigl(n+|\alpha|\bigr)^{p-2(n-j)} |a_\alpha|^2 < \infty, \tag{26}$$

by (25) for $q > 2n/p$, where $\xi_{\alpha,q} = (\int_S |\zeta^\alpha|^q \, d\sigma(\zeta))^{2/q}$, then $f_\omega \in M(\mathcal{D}_p)$ almost surely by [4, Theorem 1.1]. Therefore, $M(\mathcal{D}_p) \not\subset \mathcal{B}_n^{n-p/2}$.

(iii) The version of (iii) follows from the implication $M(\mathcal{D}_p) \subset \mathcal{D}_p$ and Theorem 1(iii) immediately. □

It is well known that $Q_p \subset \mathcal{D}_{n(1-p)}$ for $(n-1)/n < p \leqslant 1$, and $M(\mathcal{D}_{n(1-p)}) \subset \mathcal{D}_{n(1-p)}$. How about the relation of Q_p and $M(\mathcal{D}_{n(1-p)})$? Theorem 3 states that $M(\mathcal{D}_{n(1-p)})$ is a subset of Q_p for $(n-1)/n < p \leqslant 1$.

Proof of Theorem 3. For $z \in B$, let

$$g_z(w) = \left(\frac{1-|z|^2}{(1-\langle w,z\rangle)^2} \right)^{np/2}.$$

It can be verified that $g_z \in \mathcal{D}_{n(1-p)}$. In fact, since

$$\frac{\partial g_z(w)}{\partial w_j} = np\overline{z_j} \frac{(1-|z|^2)^{np/2}}{(1-\langle w,z\rangle)^{1+np}}, \tag{27}$$

then

$$\bigl|\nabla g_z(w)\bigr|^2 = \sum_{j=1}^n \left| \frac{\partial g_z(w)}{\partial w_j} \right|^2 = \frac{n^2p^2|z|^2(1-|z|^2)^{np}}{|1-\langle w,z\rangle|^{2+2np}}. \tag{28}$$

So,

$$\begin{aligned} &\int_B \bigl|\nabla g_z(w)\bigr|^2 \bigl(1-|w|^2\bigr)^{1-n(1-p)} \, d\nu(w) \\ &\quad = n^2p^2|z|^2\bigl(1-|z|^2\bigr)^{np} \int_B \frac{(1-|w|^2)^{1-n(1-p)}}{|1-\langle w,z\rangle|^{2+2np}} \, d\nu(w) \\ &\quad \leqslant Cn^2p^2, \end{aligned}$$

in the above, [1, Proposition 1.4.10] is used. Therefore, $g_z \in \mathcal{D}_{n(1-p)}$, and there is a constant K independent of z, such that $\|g_z\|_{\mathcal{D}_{n(1-p)}} \leqslant K$.

For any $f \in M(\mathcal{D}_{n(1-p)})$, [3, Corollary 1] implies that $f \in H^\infty(B)$. So

$$\begin{aligned}
&\int_B \left(\frac{1-|z|^2}{|1-\langle w,z\rangle|^2}\right)^{np} \big|\nabla f(w)\big|^2 \big(1-|w|^2\big)^{1-n(1-p)}\,d\nu(w) \\
&\quad= \int_B \big|g_z(w)\big|^2 \big|\nabla f(w)\big|^2 \big(1-|w|^2\big)^{1-n(1-p)}\,d\nu(w) \\
&\quad\leqslant C_1 \int_B \big|\nabla(fg_z)(w)\big|^2 \big(1-|w|^2\big)^{1-n(1-p)}\,d\nu(w) \\
&\qquad+ C_2 \int_B \big|f(w)\big|^2 \big|\nabla g_z(w)\big|^2 \big(1-|w|^2\big)^{1-n(1-p)}\,d\nu(w) \\
&\quad\leqslant C_1 \|fg_z\|_{\mathcal{D}_{n(1-p)}} + C_2 \int_B \big|\nabla g_z(w)\big|^2 \big(1-|w|^2\big)^{1-n(1-p)}\,d\nu(w) \\
&\quad\leqslant C_1 \|M_f\| \|g_z\|_{\mathcal{D}_{n(1-p)}} + C_2 \|g_z\|_{\mathcal{D}_{n(1-p)}} \leqslant CK.
\end{aligned}$$

Therefore

$$\sup_{z\in B} \int_B \left(\frac{1-|z|^2}{|1-\langle w,z\rangle|^2}\right)^{np} \big|\nabla f(w)\big|^2 \big(1-|w|^2\big)^{1-n(1-p)}\,d\nu(w) < \infty. \tag{29}$$

Then it follows that $f \in Q_p$ from [15, Theorems 1 and 2], (29) and the equivalence of the complex gradient and the invariant gradient. □

References

[1] W. Rudin, Function Theory in the Unit Ball of $\mathbb{C}^n$, Spinger-Verlag, New York, 1980.

[2] C. Ouyang, W. Yang, R. Zhao, Möbius invariant Q_p spaces associated with the Green's function on the unit ball of $\mathbb{C}^n$, Pacific J. Math. 182 (1998) 69–99.

[3] P. Hu, J. Shi, Multipiers on Dirichlet type spaces, Acta Math. Sinica (English Ser.) 17 (2001) 263–272.

[4] P. Hu, J. Shi, Random Dirichlet type functions on the unit ball of $\mathbb{C}^n$, Chinese Ann. Math. 20 (1999) 369–376.

[5] P. Hu, W. Zhang, A new characterization of Dirichlet type spaces on the unit ball of $\mathbb{C}^n$, J. Math. Anal. Appl. 259 (2001) 453–461.

[6] C. Ouyang, Proper inclusions of function spaces and Ryll–Wojtaszczyk polynomials on the unit ball of $\mathbb{C}^n$, in: Finite or Infinite Dimensional Complex Analysis, in: Lecture Notes in Pure and Appl. Math., vol. 214, Dekker, New York, 2000, pp. 409–426.

[7] W. Yang, C. Ouyang, Exact location of α-Bloch space in L^p_a and H^p of a complex unit ball, Rocky Mountain J. Math. 30 (2000) 1151–1169.

[8] P. Hu, W. Xu, W. Zhang, Boundedness of composition operators on $\mathcal{D}_\tau$ spaces for $\tau > 1$, Acta Math. Sci. B 20 (2000) 409–416.

[9] R.M. Timoney, Bloch functions on several complex variables, I, Bull. London Math. Soc. 12 (1980) 241–267.

[10] J. Ryll, P. Wojtaszczyk, On homogeneous polymials on a complex ball, Trans. Amer. Math. Soc. 276 (1983) 107–116.

[11] S. Yamashita, Gap series and α-bloch functions, Yokohama Math. J. 28 (1980) 31–36.

[12] R. Aulaskari, P. Lappan, J. Xiao, R. Zhao, On α-Bloch spaces and multipliers of Dirichlet spaces, J. Math. Anal. Appl. 209 (1997) 103–121.
[13] P. Hu, p-Bloch space and $Q_{p,0}$ space and the holomorphic function with Hadamard gap, J. Shenzhen Univ. (Natural Science) 19 (4) (2002) 1–9 (in Chinese).
[14] J. Shi, Hardy–Littlewood theorems on bounded symmetric domains, Scientia Sinica (Ser. A) 31 (1988) 926.
[15] W. Yang, Carleson type measure characterization of Q_p spaces, Analysis 18 (1998) 345–392.